

COMMENSURATE SCALE RELATIONS: PRECISE TESTS OF QUANTUM CHROMODYNAMICS WITHOUT SCALE OR SCHEME AMBIGUITY *

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ABSTRACT

We derive commensurate scale relations which relate perturbatively calculable QCD observables to each other, including the annihilation ratio $R_{e^+e^-}$, the heavy quark potential, τ decay, and radiative corrections to structure function sum rules. For each such observable one can define an effective charge, such as $\alpha_R(\sqrt{s})/\pi \equiv R_{e^+e^-}(\sqrt{s})/(3 \sum e_q^2) - 1$. The commensurate scale relation connecting the effective charges for observables A and B has the form $\alpha_A(Q_A) = \alpha_B(Q_B) \left(1 + r_{A/B} \frac{\alpha_B}{\pi} + \dots\right)$, where the coefficient $r_{A/B}$ is independent of the number of flavors f contributing to coupling renormalization, as in BLM scale-fixing. The ratio of scales Q_A/Q_B is unique at leading order and guarantees that the observables A and B pass through new quark thresholds at the same physical scale. In higher orders a different renormalization scale Q^{n*} is assigned for each order n in the perturbative series such that the coefficients of the series are identical to that of a conformally invariant theory. The commensurate scale relations and scales satisfy the renormalization group transitivity rule which ensures that predictions in PQCD are independent of the choice of an intermediate renormalization scheme C . In particular, scale-fixed predictions can be made without reference to theoretically constructed singular renormalization schemes such as $\overline{\text{MS}}$. QCD can thus be tested in a new and precise way by checking that the effective charges of observables track both in their relative normalization and in their commensurate scale dependence. The commensurate scale relations which relate the radiative corrections to the annihilation ratio $R_{e^+e^-}$ to the radiative corrections for the Bjorken and Gross-Llewellyn Smith sum rules are particularly elegant and interesting. The final series has simple coefficients which are independent of color: $\hat{\alpha}_{g_1}(Q) = \hat{\alpha}_R(Q^*) - \hat{\alpha}_R^2(Q^{**}) + \hat{\alpha}_R^3(Q^{***}) + \dots$, where $\hat{\alpha} = (3C_F/4\pi)\alpha$. The coeffi-

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cients coincide with the Crewther relation obtained in conformally invariant gauge theory. Thus this commensurate scale relation provides the generalization of the Crewther relation to non-conformal gauge theory.

1. Introduction

The problem of the renormalization scale dependence of perturbative QCD predictions has plagued attempts to make reliable and precise tests of the theory. There is, in fact, no consensus on how to estimate the theoretical error due to the scale ambiguity, what constitutes a reasonable range of physical values, or indeed how to identify what the central value should be. The problem is compounded in multi-scale problems where several plausible physical scales enter. Even worse, if we consider the renormalization scale as totally arbitrary, the next-to-leading coefficient in the perturbative expansion can take on the value zero or any other value. Thus it is difficult to assess the convergence of the truncated series, and finite-order analyses cannot be meaningfully compared to experiment.

Recently we have shown how the scale ambiguity problem can be avoided by focussing on relations between experimentally measurable observables [1]. The conventional $\overline{\text{MS}}$ renormalization scheme serves simply as an intermediary between observables. For example, consider the entire radiative corrections to the annihilation cross section expressed as the “effective charge” $\alpha_R(Q)$ where $Q = \sqrt{s}$:

$$R(Q) \equiv 3 \sum_f Q_f^2 \left[1 + \frac{\alpha_R(Q)}{\pi} \right]. \quad (1)$$

Similarly, we can define the entire radiative correction to the Bjorken sum rule as the effective charge $\alpha_{g_1}(Q)$ where Q is the lepton momentum transfer:¹

$$\int_0^1 dx \left[g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2) \right] \equiv \frac{1}{3} \left| \frac{g_A}{g_V} \right| \left[1 - \frac{\alpha_{g_1}(Q)}{\pi} \right]. \quad (2)$$

We now use the known expressions to three loops in $\overline{\text{MS}}$ scheme and choose the scales Q^* and Q^{**} to re-sum all quark and gluon vacuum polarization corrections into the running couplings. The values of these scales are the physical values of the energies or momentum transfers which ensure that the radiative corrections to each observable passes through the heavy quark thresholds at their respective commensurate physical scales. The final result is remarkably simple:

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \left(\frac{\alpha_R(Q^{**})}{\pi} \right)^2 + \left(\frac{\alpha_R(Q^{***})}{\pi} \right)^3 + \dots \quad (3)$$

The coefficients in the series (aside for a factor of C_F , which can be absorbed in the definition of α_s) are actually independent of color and are the same in Abelian, non-Abelian, and conformal gauge theory. The non-Abelian structure of the theory is reflected in the scales Q^* and Q^{**} .

Any perturbatively calculable physical quantity can be used to define an effective charge [2,3,4] by incorporating the entire radiative correction into its definition. An

¹Here we follow the normalization given in Ref. [5]. For a recent review on this sum rule, see Ref. [6].

important result is that all effective charges $\alpha_A(Q)$ satisfy the Gell-Mann-Low renormalization group equation with the same β_0 and β_1 ; different schemes or effective charges only differ through the third and higher coefficients of the β function. Thus, any effective charge can be used as a reference running coupling constant in QCD to define the renormalization procedure. More generally, each effective charge or renormalization scheme, including $\overline{\text{MS}}$, is a special case of the universal coupling function $\alpha(Q, \beta_n)$ [7]. Peterman and Stükelberg have shown [8] that all effective charges are related to each other through a set of evolution equations in the scheme parameters β_n .

We shall refer to the connections between the effective charges of observables such as Eq. (3) as “commensurate scale relations” (CSR) [1]. A fundamental test of QCD will be to verify empirically that the observables track in both normalization and shape as given by the CSR. The commensurate scale relations thus provide fundamental tests of QCD which can be made increasingly precise and independent of the choice of renormalization scheme or other theoretical convention.

The relation between physical observables must be independent of the choice of any intermediate renormalization scheme. In a renormalizable quantum field theory, such as quantum chromodynamics, the Lagrangian $\mathcal{L}[g(\mu), m_i(\mu), \dots]$ is written a function of the coupling $g(\mu)$ and masses $m_i(\mu)$ at a given scale μ . In principle, one can choose a minimal set of input measurements to fix these bare parameters, and then predict all other physical observables. For example, in quantum electrodynamics, it is conventional to use Coulomb scattering $\ell\ell \rightarrow \ell\ell$ extrapolated to zero momentum transfer and threshold Compton scattering $\gamma\ell \rightarrow \gamma\ell$ to fix the bare charge and lepton mass parameters. One then can systematically predict the values of the lepton anomalous magnetic moments and the other high precision QED observables. Renormalization group equations reflect the fact that predictions connecting the input and output observables can be computed without ambiguity order by order in perturbation theory; *i.e.*, the relations connecting physical observables are independent of the choice of the scale μ and the choice of intermediate renormalization scheme.

The theoretical predictions connecting any pair of perturbatively calculable observables are given explicitly by the commensurate scale relations. In these relations between the effective charges of the physical observables, the dependence on the value of the bare coupling and the choice of renormalization procedure cancels out, as expected, order by order in perturbation theory. Thus the CSR provide precise and direct experimental tests of quantum field theory without scale or scheme ambiguity.

The CSR for observables A and B in terms of their effective charges has the form

$$\alpha_A(Q_A) = \alpha_B(Q_B) \left(1 + r_{A/B} \frac{\alpha_B}{\pi} + \dots \right). \quad (4)$$

The ratio of the scales $\lambda_{A/B} = Q_A/Q_B$ is fixed by the requirement that the coefficient $r_{A/B}$ is independent of the number of flavors f contributing to coupling constant renormalization. This guarantees that the effective charges for the observables A and B pass through new quark thresholds at the same physical scale. The scales Q_A and Q_B are thus commensurate. The value of $\lambda_{A/B}$ is unique at leading order, and the relative scales satisfy the transitivity rule [9]

$$\lambda_{A/B} = \lambda_{A/C} \lambda_{C/B}. \quad (5)$$

This is equivalent to the group property defined by Peterman and Stükelberg [8] which ensures that predictions in PQCD are independent of the choice of an interme-

diate renormalization scheme C^2 (The renormalization group method was developed by Gell-Mann and Low [10] and by Bogoliubov and Shirkov [11].) In particular, scale-fixed predictions can be made without reference to theoretically constructed renormalization schemes such as $\overline{\text{MS}}$; QCD can thus be tested by checking that the observables track both in their relative normalization and commensurate scale dependence [12].

The transitivity and symmetry properties of the commensurate scales are the scale transformations of the renormalization “group” as originally defined by Peterman and Stückelberg [8]. The predicted relation between observables must be independent of the order one makes substitutions; *i.e.* the algebraic path one takes to relate the observables. Furthermore, any method which fixes the scale in QCD must also be applicable to Abelian theories such as QED, since in the limit of small number of colors $N_C \rightarrow 0$ or large number of flavors f the perturbative coefficients in QCD coincide with the perturbative coefficients of an Abelian analog of QCD³.

The relation between scales in the CSR is consistent with the BLM scale-fixing procedure [13] in which the scale is chosen such that all terms arising from the QCD β -function are resummed into the coupling. Note that this also implies that the coefficients in the perturbation CSR expansions are independent of the number of quark flavors f renormalizing the gluon propagators. This prescription ensures that, as in quantum electrodynamics, vacuum polarization contributions due to fermion pairs are all incorporated into the coupling $\alpha(\mu)$ rather than the coefficients.

One of the most useful observables in QCD is the heavy quark potential, since it can be computed in lattice gauge theory from a Wilson loop. Since interacting quarks have infinite mass, the radiative correction are associated with the exchange diagrams, rather than the vertex corrections. It is convenient to write the heavy quark potential as $V(Q^2) = -4\pi C_F \alpha_V(Q)/Q^2$. This defines the effective charge $\alpha_V(Q^2)$ where by definition the “self-scale” $Q^2 = -t$ is the momentum transfer squared.

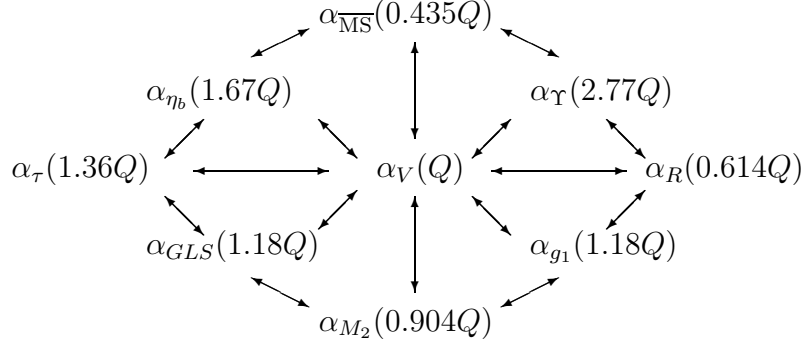
The BLM scale has also recently been used by Lepage and Mackenzie [14] and their co-workers to improve lattice perturbation theory. By using the BLM method one can eliminate α_{Lattice} in favor of α_V thus avoiding an expansion with artificially large coefficients. An essential step in this derivation is the use of the BLM method to connect the scale π/a in lattice perturbation theory to the physical scale appearing in α_V . The lattice determination, together with the empirical constraints from the heavy quarkonium spectra, promises to provide a well-determined effective charge $\alpha_V(Q)$ which could be adopted as the QCD standard coupling. In fact, a precise determination $\alpha_V(8.2 \text{ GeV}) = 0.1957(34)$ has recently been obtained (including 3 dynamical fermions) from the Υ spectrum by the Cornell lattice group [15].

In the case of non-Abelian theory, the BLM method automatically resums the corresponding gluon as well as quark vacuum polarization contributions since the coupling α_s is a function of $\beta_0 \propto 11 - \frac{2}{3}f$. For example in the α_V scheme the coupling is defined to sum all vacuum polarization contribution, so that coefficients of the expansion in powers of this coupling cannot depend on the number of flavors f arising from vacuum polarization. The transitivity property of the renormalization group equations then requires that this is true when expanding in any effective charge.

²We thank A. Kataev for an illuminating discussion on this point.

³We thank Patrick Huet and Eric Sather for conversations on this point.

Table I
Leading Order Commensurate Scale Relations



Alternatively, one can derive the leading-order BLM scale when expanding in α_V by explicitly integrating the one loop integrals in the calculation of the observable A using $\alpha_V(\ell^2)$ in the integrand, where ℓ^2 is the four-momentum transferred squared carried by the gluon. (In practice one only needs to compute the mean-value $\langle \ln \ell^2 \rangle = \ln Q_V^2$ [14].)

In general, the coefficients in the perturbative expansion using BLM scale fixing are the same as those of the corresponding conformally invariant theory with $\beta = 0$. In practice, the conformal limit is defined by $\beta_0, \beta_1 \rightarrow 0$, and can be reached, for instance, by adding enough spin-half and scalar quarks as in supersymmetric QCD with $N_C = 4$. Since all the running coupling effects have been absorbed into the renormalization scales, the scale setting method described here correctly reproduces the expansion coefficients in this limit.⁴ It should be pointed out that other scale-setting procedures do not guarantee this feature.

The scale-fixed relation between the heavy quark potential effective charge and $\alpha_{\overline{\text{MS}}}$ is $\alpha_V(Q) = \alpha_{\overline{\text{MS}}}(e^{-5/6}Q)[1 - 2(\alpha_{\overline{\text{MS}}}/\pi) + \dots]$ [13]. (The one-loop calculation of α_V in $\overline{\text{MS}}$ scheme is given in Ref. [16].) The new result of Ref. [15] then implies $\alpha_{\overline{\text{MS}}}^{(3)}(3.56 \text{ GeV}) = 0.2201(84)$ and a very precise value at $Q = M_z$: $\alpha_{\overline{\text{MS}}}^{(5)}(M_z) = 0.115(2)$. This commensurate scale relation provides an analytic definition of $\overline{\text{MS}}$ scheme which automatically incorporates the transition between quark flavors.

The physical value of the commensurate scale in α_V scheme reflects the mean virtuality of the exchanged gluon. However, in other schemes, including $\overline{\text{MS}}$, the argument of the effective charge is displaced from its physical value. The relative scale for a number of observables is indicated in Table I. For example, the physical scale for the branching ratio $\Upsilon \rightarrow \gamma X$ when expanded in terms of α_V is $(1/2.77)M_\Upsilon \sim (1/3)M_\Upsilon$, which reflects the fact that the final state phase space is divided among three vector systems. (When one expands in $\overline{\text{MS}}$ scheme, the corresponding scale is $0.157M_\Upsilon$.) Similarly, the physical scale appropriate to the hadronic decays of the η_b is $(1/1.67)M_{\eta_b} \sim (1/2)M_{\eta_b}$.

⁴We thank Dieter Müller for discussions on this point.

The BLM method has recently been applied to the analysis of jet ratios in e^+e^- collisions by Kramer and Lampe [17] and jet ratios in ep collisions by Ingelman and Rathsman [18]. One can determine the scale Q^* for $(2+1)$ jets at HERA as a function of all of the available scales. The method has also been applied to the radiative corrections to the top width decay by Voloshin and Smith [19] and to other electroweak measures by Sirlin [20].

2. CSR in Higher Order

It is straightforward to derive the general connection between any two observables in QCD through order α_s^3 if one is given the calculation of each of the observables to NNLO in $\overline{\text{MS}}$ scheme. As we have emphasized, the choice of the intermediate scheme is irrelevant for the final CSR connecting the two observables.

The first step is to eliminate the intermediate scheme algebraically. The expansion series of a physical effective charge $\alpha_1(Q)/\pi$ in terms of another physical effective charge $\alpha_2(Q)/\pi$ has the form [1]

$$\begin{aligned} \frac{\alpha_1(Q)}{\pi} &= \frac{\alpha_2(Q)}{\pi} + (A_{12} + B_{12}f) \left(\frac{\alpha_2(Q)}{\pi} \right)^2 \\ &+ (C_{12} + D_{12}f + E_{12}f^2) \left(\frac{\alpha_2(Q)}{\pi} \right)^3 + \dots \end{aligned} \quad (6)$$

After the resummation of running coupling effects into a set of new renormalization scales Q^* , Q^{**} and Q^{***} , one arrives at a series of the form⁵

$$\frac{\alpha_1(Q)}{\pi} = \frac{\alpha_2(Q^*)}{\pi} + \tilde{A}_{12} \left(\frac{\alpha_2(Q^{**})}{\pi} \right)^2 + \tilde{C}_{12} \left(\frac{\alpha_2(Q^{***})}{\pi} \right)^3 + \dots \quad (7)$$

$$\tilde{A}_{12} = A_{12} + \frac{11}{4} \frac{C_A}{T} B_{12}, \quad (8)$$

$$\tilde{C}_{12} = -\frac{3}{16} \frac{C_A}{T} (7C_A + 11C_F) B_{12} + C_{12} + \frac{11}{4} \frac{C_A}{T} D_{12} + \frac{121}{16} \frac{C_A^2}{T^2} E_{12}, \quad (9)$$

⁵We must exclude from the analysis the potential f dependence in the NNLO term induced by light-by-light diagrams. These diagrams are finite and do not participate in the renormalization of the running coupling constant. As a convention, the coefficient D_{12} in Eq. (6) will include only the f dependence from the running of the coupling constant. The extra f -dependent terms from light-by-light scattering diagrams will be considered as part of the C_{12} . This separation is straightforward in the practical examples considered here.

$$Q^* = Q \exp \left[\frac{3}{2T} B_{12} + \frac{9}{8T^2} \left(\frac{11}{3} C_A - \frac{4}{3} T f \right) (B_{12}^2 - E_{12}) \frac{\alpha_2(Q)}{\pi} \right], \quad (10)$$

$$Q^{**} = Q \exp \left\{ \frac{3}{4T} \tilde{A}_{12}^{-1} \left[-\frac{1}{4} (5C_A + 3C_F) B_{12} + D_{12} + \frac{11}{2} \frac{C_A}{T} E_{12} \right] \right\}. \quad (11)$$

Notice the presence of $\alpha_2(Q)/\pi$ in the expression of Q^* . In general Q^* will itself be a perturbative series in $\alpha_2(Q)/\pi$. We have exponentiated the perturbative series, since physically, the renormalization scale Q^* should always be positive. To the order considered here, the scale for the coupling constant in Eq. (10) is not well-defined, but can be chosen to be Q . This intrinsic uncertainty is similar to the Q^{***} scale uncertainty of the NNLO term in Eq. (7), and can only be resolved by going to the next-higher order.

As an illustrative example, we quote the perturbative series of $\alpha_{g_1}(Q)/\pi$ using dimensional regularization and the $\overline{\text{MS}}$ scheme with the renormalization scale fixed at $\mu = Q$: [5]

$$\begin{aligned} \frac{\alpha_{g_1}(Q)}{\pi} &= \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \right)^2 \left[\frac{23}{12} C_A - \frac{7}{8} C_F - \frac{1}{3} f \right] \\ &+ \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \right)^3 \left\{ \left(\frac{5437}{648} - \frac{55}{18} \zeta_5 \right) C_A^2 + \left(-\frac{1241}{432} + \frac{11}{9} \zeta_3 \right) C_A C_F + \frac{1}{32} C_F^2 \right. \\ &\quad + \left[\left(-\frac{3535}{1296} - \frac{1}{2} \zeta_3 + \frac{5}{9} \zeta_5 \right) C_A + \left(\frac{133}{864} + \frac{5}{18} \zeta_3 \right) C_F \right] f \\ &\quad \left. + \frac{115}{648} f^2 \right\}. \end{aligned} \quad (12)$$

The effective charge for the annihilation cross section has been computed in the $\overline{\text{MS}}$ scheme with the renormalization scale fixed at $\mu = Q = \sqrt{s}$ [21,22]. The perturbative series for $\alpha_R(Q)/\pi$ is (using $T = 1/2$ for the trace normalization)

$$\begin{aligned} \frac{\alpha_R(Q)}{\pi} &= \frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \right)^2 \left[\left(\frac{41}{8} - \frac{11}{3} \zeta_3 \right) C_A - \frac{1}{8} C_F + \left(-\frac{11}{12} + \frac{2}{3} \zeta_3 \right) f \right] \\ &+ \left(\frac{\alpha_{\overline{\text{MS}}}(Q)}{\pi} \right)^3 \left\{ \left(\frac{90445}{2592} - \frac{2737}{108} \zeta_3 - \frac{55}{18} \zeta_5 - \frac{121}{432} \pi^2 \right) C_A^2 \right. \\ &\quad + \left(-\frac{127}{48} - \frac{143}{12} \zeta_3 + \frac{55}{3} \zeta_5 \right) C_A C_F - \frac{23}{32} C_F^2 \\ &\quad + \left[\left(-\frac{970}{81} + \frac{224}{27} \zeta_3 + \frac{5}{9} \zeta_5 + \frac{11}{108} \pi^2 \right) C_A \right. \\ &\quad \left. + \left(-\frac{29}{96} + \frac{19}{6} \zeta_3 - \frac{10}{3} \zeta_5 \right) C_F \right] f \\ &\quad + \left(\frac{151}{162} - \frac{19}{27} \zeta_3 - \frac{1}{108} \pi^2 \right) f^2 \\ &\quad \left. + \left(\frac{11}{144} - \frac{1}{6} \zeta_3 \right) \frac{d^{abc} d^{abc}}{C_F d(R)} \frac{(\sum_f Q_f)^2}{\sum_f Q_f^2} \right\}. \end{aligned} \quad (13)$$

The term containing $(\sum_f Q_f)^2 / \sum_f Q_f^2$ arises from light-by-light diagrams. The dimension of the quark representation is $d(R)$, which usually is N for $SU(N)$. For QCD we have $d^{abc}d^{abc} = 40/3$.

The application of the NLO BLM formulas then leads to

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \frac{3}{4}C_F \left(\frac{\alpha_R(Q^{**})}{\pi} \right)^2 + \left[\frac{9}{16}C_F^2 - \left(\frac{11}{144} - \frac{1}{6}\zeta_3 \right) \frac{d^{abc}d^{abc}}{C_F N} \frac{(\sum_f Q_f)^2}{\sum_f Q_f^2} \right] \left(\frac{\alpha_R(Q^{***})}{\pi} \right)^3, \quad (14)$$

$$Q^* = Q \exp \left[\frac{7}{4} - 2\zeta_3 + \left(\frac{11}{96} + \frac{7}{3}\zeta_3 - 2\zeta_3^2 - \frac{\pi^2}{24} \right) \left(\frac{11}{3}C_A - \frac{2}{3}f \right) \frac{\alpha_R(Q)}{\pi} \right], \quad (15)$$

$$Q^{**} = Q \exp \left[\frac{523}{216} + \frac{28}{9}\zeta_3 - \frac{20}{3}\zeta_5 + \left(-\frac{13}{54} + \frac{2}{9}\zeta_3 \right) \frac{C_A}{C_F} \right]. \quad (16)$$

(The scale Q^{***} in the above expression can be chosen to be Q^{**} .) Notice that aside from the light-by-light contributions, all the ζ_3, ζ_5 and π^2 dependencies have been absorbed into the renormalization scales Q^* and Q^{**} . Understandably, the π^2 term should be absorbed into renormalization scale since it comes from the analytical continuation of $R(Q)$ to the Euclidean region.

For the three flavor case, where we can neglect the light-by-light contribution, the series remarkably simplifies to the CSR of Eq. (3). The form suggest that for the general $SU(N)$ group the natural expansion parameter is $\hat{\alpha} = (3C_F/4\pi)\alpha$. The use of $\hat{\alpha}$ also makes explicit that the same formula is valid for QCD and QED. That is, in the limit $N_C \rightarrow 0$ the perturbative coefficients in QCD coincide with the perturbative coefficients of an Abelian analog of QCD.

3. Commensurate Scale Relations and the Crewther Relation

Broadhurst and Kataev have recently observed a number of interesting relations between $\alpha_R(Q)$ and $\alpha_{g_1}(Q)$ (the “Seven Wonders”) [23]. In particular, they have shown the factorization of the beta function in the correction to the Crewther’s relation [24] which establishes a non-trivial connection between the total e^+e^- annihilation cross section and the polarized Bjorken sum rule. The simple form of our result Eq. (3) also points to the existence of a “secret symmetry” between $\alpha_R(Q)$ and $\alpha_{g_1}(Q)$ which is revealed after the application of the NLO BLM scale setting procedure. In fact, as pointed out by Kataev and Broadhurst [23], in the conformally invariant limit, *i.e.*, for vanishing beta functions, Crewther’s relation becomes

$$(1 + \hat{\alpha}_R^{\text{eff}})(1 - \hat{\alpha}_{g_1}^{\text{eff}}) = 1. \quad (17)$$

which is equivalent to our result in Eq. (3). Thus Eq. (3) can be regarded as the extension of the Crewther relation to non-conformally invariant gauge theory.

The commensurate scale relation between α_{g_1} and α_R given by Eq. (14) implies that the radiative corrections to the annihilation cross section and the Bjorken (or

Gross-Llewellyn Smith) sum rule cancel. The relation between the physical cross sections can be written in the forms:

$$\frac{R_{e^+e^-}(s)}{3 \sum e_q^2} \frac{\int_0^1 dx g_1^p(x, Q^2) - g_1^n(x, Q^2)}{\frac{1}{3} g_A / g_V} = 1 - \Delta \beta_0 \hat{a}^3 \quad (18)$$

and

$$\frac{R_{e^+e^-}(s)}{3 \sum e_q^2} \frac{\int_0^1 dx F_3^{\nu p}(x, Q^2) + F_3^{\bar{\nu} p}(x, Q^2)}{6} = 1 + \Delta \beta_0 \hat{a}^3, \quad (19)$$

provided that the annihilation energy in $R_{e^+e^-}(s)$ and the momentum transfer Q appearing in the deep inelastic structure functions are commensurate at NLO: $\sqrt{s} = Q^* = Q \exp[\frac{7}{4} - 2\zeta_3 + (\frac{11}{96} + \frac{7}{3}\zeta_3 - 2\zeta_3^2 - \frac{\pi^2}{24})\beta_0 \hat{a}(Q)]$

The light-by-light correction to the CSR for the Bjorken sum rule vanishes for three flavors. The term $\Delta \beta_0 \hat{a}^3$ with $\Delta = \ell n(Q^*/Q^{**})$ is the third-order correction arising from the difference between Q^{**} and Q^* ; in practice this correction is negligible: for a typical value $\hat{a} = \alpha_R(Q)/4\pi = 0.14$, $\Delta \beta_0 \hat{a}^3 = 0.00071$. Thus at the magic energy $\sqrt{s} = Q^*$, the radiative corrections to the Bjorken and GLLS sum rules almost precisely cancel the radiative corrections to the annihilation cross section. This allows a practical test and extension of the Crewther relation to non-conformal QCD.

As an initial test of this CSR, we can compare the recent CCHS measurement of the Gross-Llewellyn Smith sum rule $1 - \hat{\alpha}_{F_3} = \frac{1}{6} \int_0^1 dx [F_3^{\nu p}(x, Q^2) + F_3^{\bar{\nu} p}(x, Q^2)] = \frac{1}{3}(2.5 \pm 0.13)$ at $Q^2 = 3 \text{ GeV}^2$ and the parameterization of the annihilation data [25] $1 + \hat{\alpha}_R = R_{e^+e^-}(s)/3 \sum e_q^2 = 1.20$. at the commensurate scale $\sqrt{s} = Q^* = 0.38 Q = 0.66 \text{ GeV}$. The product is $(1 + \hat{\alpha}_R)(1 - \hat{\alpha}_{F_3}) = 1.00 \pm 0.04$, which is a highly nontrivial check of the theory at very low physical scales.

4. Other Applications of CSR

As another example of a beyond-leading-order commensurate scale relation, we shall express the effective charge for τ decay $\alpha_\tau(M_\tau)/\pi$ in terms of the effective charge for e^+e^- annihilation $\alpha_R(Q)/\pi$. The appropriate number of flavors in this case is $f = 3$, because τ decay occurs below the charm threshold. [Incidentally, the light-by-light contribution in $\alpha_R(Q)/\pi$ vanishes for the three flavor case.] The application of the NLO BLM formulas leads to the following commensurate scale relation

$$\frac{\alpha_\tau(M_\tau)}{\pi} = \frac{\alpha_R(Q^*)}{\pi}, \quad (20)$$

$$Q^* = M_\tau \exp \left[-\frac{19}{24} - \frac{169}{128} \frac{\alpha_R(M_\tau)}{\pi} \right]. \quad (21)$$

Notice that all the ζ_3, ζ_5 and π^2 terms present in the perturbative series of $\alpha_R(Q)/\pi$ and $\alpha_\tau(M_\tau)/\pi$ have disappeared when we related these two physical observables directly. Notice also the vanishing NLO and NNLO coefficient in Eq. (20). That is, up to the NNLO, the two effective charge are simply related by a BLM scale shift.

Since the radiative corrections to the Bjorken sum rule are identical to those of the Gross-Llewellyn-Smith sum rule—up to small corrections of order $\alpha_s^3(Q^2)$, a basic test of QCD can be made by considering the ratio of the Gross-Llewellyn-Smith and Bjorken sum rules:

$$R_{GLLS/Bj}(Q^2, \epsilon) = \frac{\frac{1}{6} \int_{\epsilon}^1 dx [F_3^{\nu p}(x, Q^2) + F_3^{\bar{\nu} p}(x, Q^2)]}{3 \left| \frac{g_V}{g_A} \right| \int_{\epsilon}^1 dx [g_1^p(x, Q^2) - g_1^n(x, Q^2)]}. \quad (22)$$

If the Regge behavior of the two sum rules is similar, the empirical extrapolation to $\epsilon \rightarrow 0$ should be relatively free of systematic error. Moreover, PQCD predicts

$$R_{GLLS/Bj}(Q^2, \epsilon \rightarrow 0) = 1 + \mathcal{O}(\alpha_s^3(Q)) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{Q^2}\right), \quad (23)$$

i.e., hard relativistic corrections to the ratio of the sum rules only enter at three loops. Thus measurements of the *ratio* of the sum rules could provide a remarkably complication-free test of QCD - any significant deviation from $R_{GLLS/Bj}(Q^2, \epsilon \rightarrow 0) = 1$ must be due to higher twist effects which should vanish rapidly with increasing Q^2 .

After scale-fixing, the ratio of hadronic to leptonic decay rates for the Υ has the form [13]

$$\frac{\Gamma(\Upsilon \rightarrow \text{hadrons})}{\Gamma(\Upsilon \rightarrow \mu^+ \mu^-)} = \frac{10(\pi^2 - 9)}{81\pi e_b^2} \frac{\alpha_{\overline{\text{MS}}}^3(0.157M_{\Upsilon})}{\alpha_{\text{QED}}^2} \left[1 - 14.0(5) \frac{\alpha_{\overline{\text{MS}}}}{\pi} + \dots \right] \quad (24)$$

$$= \frac{10(\pi^2 - 9)}{81\pi e_b^2} \frac{\alpha_V^3(0.363M_{\Upsilon})}{\alpha_{\text{QED}}^2} \left[1 - 8.0(5) \frac{\alpha_V}{\pi} + \dots \right]. \quad (25)$$

Thus, as is the case of positronium decay, the next to leading coefficient is very large, and perturbation theory is not likely to be reliable for this observable. On the other hand, the commensurate scales for the second moment of the non-singlet structure function M_2 and the effective charges in the Bjorken Sum Rule (and the Gross-Llewellyn-Smith Sum Rule) are not far from the physical value Q when expressed in α_V scheme. At large n the commensurate scale for M_n is proportional to $1/\sqrt{n}$, reflecting the fact that the available phase-space for parton emission decreases as n increases. In multiple-scale cases, the commensurate scale can depend on all of the physical invariants. For example, the scale controlling the evolution equation for the non-singlet structure function depends on x_{Bj} as well as Q [26]. In the case of inclusive reactions which factorize at leading twist, each structure function, fragmentation function, and subprocess cross section can have its own scale.

After one fixes the renormalization scale μ to the BLM value, it is still useful to compute the logarithmic derivative of the truncated perturbative prediction $d \ln \rho_N / d \ln \mu$ for a physical observable ρ at the BLM-determined scale. If this derivative is large, or equivalently, if the BLM and PMS [27] scales strongly differ, then one knows that the truncated perturbative expansion cannot be numerically reliable, since the entire series is independent of μ . Note that this is a necessary condition for a reliable series, not a sufficient one, as evidenced by the large coefficients in the positronium and quarkonium decay widths which appear when the scales are set correctly. In the case of the two and three jet decay fractions in e^+e^- annihilation, the BLM and PMS scales strongly differ at low values of the jet discriminant y . Thus,

by using this criterion, we establish that perturbation theory must fail in the small y regime, requiring careful resummation of the $\alpha_s \ln y$ series. (A more detailed discussion of the sensitivity of the jet fractions to scale choice and jet clustering schemes is given in Ref. [28].)

However, if we restrict the analysis to jets with invariant mass $\mathcal{M} < \sqrt{ys}$, with $0.14 > y > 0.05$, then we have an ideal situation, since both the PMS and FAC scales nearly coincide with the BLM scale when one computes jet ratios in the $\overline{\text{MS}}$ scheme *i.e.*, the renormalization scale dependence in this case is minimal at the BLM scale, and the computed NLO (next-to-leading order) coefficient is nearly zero. In fact, Kramer and Lampe [17] find that the BLM scale and the NLO PQCD predictions give a consistent description of the LEP 2-jet and 3-jet data for $0.14 > y > 0.05$ at the Z . Neglecting possible uncertainties due to hadronization effects, this allows a determination of α_s with remarkably small error: [17] $\alpha_{\overline{\text{MS}}}(M_Z) = 0.107 \pm 0.003$, which corresponds to $\Lambda_{\overline{\text{MS}}}^{(5)} = 100 \pm 20$ MeV. It is clear that reanalyses of the SLD and LEP data will need to be done with BLM scale breaking.

5. Conclusions

The commensurate scale relations open up additional possibilities for testing QCD. One can compare two observables by checking that their effective charges agree both in normalization and in their scale dependence. The ratio of commensurate scales $\lambda_{A/B}$ is fixed uniquely: it ensures that both observables A and B pass through heavy quark thresholds at precisely the same physical point. The same procedure can be applied to multi-scale problems; in general, the commensurate scales Q^* , Q^{**} , etc. will depend on all of the available scales.

Calculations are often performed most advantageously in $\overline{\text{MS}}$ scheme, but all reference to such theoretically constructed schemes have to vanish when comparisons are made between observables. We emphasize that any consistent renormalization scheme, with any arbitrary choice of renormalization scale μ , can be used in the intermediate stages of analysis. The final result, the commensurate scale relation between observables, is guaranteed to be independent of the choice of intermediate renormalization scheme since the BLM procedure satisfies the generalized renormalization group properties of Peterman and Stuckelberg. An important computational advantage is that one only needs to compute the f -dependence of the higher order terms in order to specify the lower order scales in the commensurate scale relations. In many cases, the series coefficients in the commensurate scale relations can be determined from the corresponding Abelian theory; *i.e.* $N_C \rightarrow 0$.

The BLM method and the commensurate scale relations presented here can be applied to the whole range of QCD and standard model processes, making the tests of theory much more sensitive. The method should also improve precision tests of electroweak, supersymmetry and other non-Abelian theories. One of the most interesting and important areas of application of commensurate scale relations will be to the hadronic corrections to exclusive and inclusive weak decays of heavy quark systems, since the scale ambiguity in the QCD radiative corrections is at present often the largest component in the theoretical error entering electroweak phenomenology.

We have also presented in this talk a number of other commensurate scale relations using the extension of the BLM method to the next-to-leading order. We have shown that in each case the application of the NLO BLM formulas to relate known physical

observables in QCD leads to results with surprising elegance and simplicity. The commensurate scale relations for some of the observables ($\alpha_R, \alpha_\tau, \alpha_{q_1}$ and α_{F_3}) are universal in the sense that the coefficients of $\hat{\alpha}_s$ are independent of color; in fact, they are the same as those for Abelian gauge theory. Thus much information on the structure of the non-Abelian commensurate scale relations can be obtained from much simpler Abelian analogs. In fact, in the examples we have discussed here, the non-Abelian nature of gauge theory is reflected in the β -function coefficients and the choice of second-order scale Q^{**} .

Because they relate observables to observables, the commensurate scale relations are convention-independent; *i.e.*, independent of the normalization conventions used to define the color $SU(N)$ matrices, etc. Since the ambiguities due to scale and scheme choice have been eliminated, one can ask fundamental questions concerning the nature of the QCD perturbative expansions, *e.g.*, whether the series is convergent or asymptotic, due to renormalons, etc. [29].

The commensurate scale relations between observables can be tested at quite low momentum transfers, even where PQCD relationships would be expected to break down. It is possible that some of the higher twist contributions common to the two observables are also correctly represented by the commensurate scale relations. In contrast, expansions of any observable in $\alpha_{\overline{\text{MS}}}(Q)$ must break down at low momentum transfer since $\alpha_{\overline{\text{MS}}}(Q)$ becomes singular at $Q = \Lambda_{\overline{\text{MS}}}$. (For example, in the 't Hooft scheme where the higher order $\beta_n = 0$ for $n = 2, 3, \dots$, $\alpha_{\overline{\text{MS}}}(Q)$ has a simple pole at $Q = \Lambda_{\overline{\text{MS}}}$.) The commensurate scale relations allow tests of QCD in terms of finite effective charges without explicit reference to singular schemes such as $\overline{\text{MS}}$. The coefficients in the CSR are identical to the coefficients in a conformal theory where renormalons do not appear. It is thus reasonable to expect that the series expansions appearing in the CSR are convergent when one relates finite observables to each other. Thus commensurate scale relations between observables allow tests of perturbative QCD with higher and higher precision as the perturbative expansion grows.

A natural procedure for developing a precision QCD phenomenology is to choose one effective charge as the canonical definition of the QCD coupling, and then predict all other observables in terms of this canonical measure. Ideally, the heavy quark effective charge $\alpha_V(Q^2)$ could serve this central role since it can be determined from both the quarkonium spectrum and from lattice gauge theory. However, it will be necessary to compute the relation of the heavy quark potential to other schemes through three loops. At present, the most precisely theoretically and empirically known effective coupling is $\alpha_R(Q^2)$, as determined from the annihilation cross section; thus it is natural to use it as the standard definition. Alternatively, one can follow historical convention and continue to use the $\overline{\text{MS}}$ scheme as an intermediary between observables. For definiteness, let us consider a 't Hooft scheme with $\Lambda = \Lambda_{\overline{\text{MS}}}$ having all $\beta_n = 0$ beyond $n = 1$. The commensurate scale relations such as Eq. (10) and (11) then unambiguously specify all of the scales Q^*, Q^{**} , etc. required to relate $\alpha_{\overline{\text{MS}}}$ to the observables. The intrinsic QCD scale will then be unambiguously encoded as $\Lambda_{\overline{\text{MS}}}$. However, as we have emphasized, there is an intrinsic disadvantage in using $\alpha_{\overline{\text{MS}}}(Q)$ as an expansion parameter: the function $\alpha_{\overline{\text{MS}}}(Q)$ has a simple pole at $Q = \Lambda_{\overline{\text{MS}}}$, whereas observables are by definition finite.

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